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Photon-number distributions in the steady states of traditional and random lasers

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Abstract : The theory of a laser has provided many semiclassical and quantum models of the laser. Differential-difference equations are used in the description of the dynamics of the laser. Focus on steady state leads to a simplification of these equations to the difference equations. In study of random lasers, traditional laser characteristics randomize, and these variables themselves are characterized after a judicious choice of a model.

Keywords : Semiclassical laser model, steady state photon statistics

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1. Introduction

After the advent of laser [1,2], this phenomenon has been studied thoroughly [3,4]. Of many models of the traditional laser it is proper to remind of the semiclassical description by three ordinary differential equations for the complex amplitude of the laser field (in photon-number units), the polarization amplitude, and the inversion (the difference between the upper-level and lower-level populations). The thermodynamic limit of the fully quantum mechanical description has been studied, *e.g.* [5]. The semiclassical description using two (rate) equations is appropriate to a semiconductor laser in which carrier-carrier scattering damps the polarization rapidly. The polarization may therefore be adiabatically eliminated [6]. The equations become

$$\frac{d}{dt}n = -gn + K(n+1)N, \tag{1}$$

$$\frac{d}{dt}N = P - aN - K(n+1)N, \tag{2}$$

where n is the complex amplitude, N is the inversion, g is the photon decay rate, a is a nonradiative loss rate, P is the pumping rate, and K is the branching ratio. Rice and Carmichael [6] studied the steady-state solution to the rate eqs. (1), (2) in dependence on the scaled pumping rate P/g and the scaled branching ratio K/g. They have carried out numerical calculations for $g = a + K$, $p/g = 10^{-1}, \ldots, 10^{8}$, $K/g = 10^{-6}, \ldots, 1$. For $K/g = 1$ ($a = 0$) they speak of the cavity-QED limit, where the laser is a thresholdless device, and, for $K/g \to 0$, of the 'thermodynamic' limit, where the concept of laser threshold is well-defined. They go to a probabilistic description and study behaviour of the Fano factor as a characteristic of fluctuations of the photon number.

Misirpashaev and Beenakker [7] study a multimode laser with a chaotic cavity. In it a mechanism of multimode generation can be assumed. Randomness of coefficients of the differential equations describing the numbers of photons in the modes and the density of population inversion leads to randomness of the number of excited modes, whose mean has been found by these authors.

Patra [8] studies the multimode laser with a chaotic cavity as well, but he considers steady-state fluctuations

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of the photon number of each sample. He arrives at a conclusion that the Fano factor increases dependent on the number of lasing modes above the threshold.

2. Model of traditional laser

The equations of population dynamics can be applied in many fields [9]. They appear in the model for random laser [8], but completed with Langevin terms, which corresponds to a stochastic dynamics. As usual, the joint probability distribution $p(n_1,\ldots,n_{N_p},N_1\ldots,N_{N_s},t)$ of photon numbers n_j, $j = 1,\ldots, N_p$, and of densities of excited atoms N_k, $k = 1,\ldots, N_s$, obeys rate equations, *i.e.* a differential-difference equation and an initial condition at $t = t_0$. The steady state of the process, *i.e.* the joint probability distribution of the steady state, *i.e.* a limit of those for $t_0 \rightarrow \infty$ obeys a difference equation.

In this paper, we assume $N_p = N_s = 1$, $a = 0$, and will deal with a differential-difference equation

$$\frac{\partial}{\partial t}p(n,N,t) = L_{\mathrm{att}}p(n,N,t) + L_{\mathrm{amp}}p(n,N,t) + L_{\mathrm{nln}}p(n,N,t), \quad (3)$$

where

$$L_{\mathrm{att}}p(n,N,t) = g[(n+1)p(n+1,N,t) - np(n,N,t)], \quad (4)$$

$$L_{\mathrm{amp}}p(n,N,t) = P[p(n, N-1, t) - p(n, N,t)], \quad (5)$$

$$L_{\mathrm{nln}}p(n,N,t) = K[n(N+1)p(n-1, N+1, t) - (n+1)Np(n,N,t)]. \quad (6)$$

For this equation, we solve an initial-value problem

$$p(n,N,t)\big|_{t=t_0} = p_0(n,N,t_0). \quad (7)$$

Eq. (3) has the property of rate equations, *i.e.* if the initial condition (7) represents the probability distribution, the solution is a probability distribution. Let us recall that $p(n,N,t)$ ($p_0(n,N,t_0)$) is a probability distribution if and only if

$$p(n,N,t) \geq 0, \ \sum_{n=0}^{\infty}\sum_{N=0}^{\infty} p(n,N,t) = 1 \quad (8)$$

and similarly for $p_0(n,N,t_0)$.

3. Treatment of integrated intensities

When $g = 0$, the existence (uniqueness) of solution for $t \in (t_0,\infty]$ is obvious. Similarly for $g > 0$, but $K = 0$, we succeed with the method of Poisson transformation. For illustration we perform this transformation in a physical fashion, *i.e.* for $K > 0$ and both arguments n and N. The variables n, N, which are numbers of particles, are replaced by the integrated intensities w, W.

The Poisson transform of a function $P(w,W,t)$ is

$$p(n,N,t) = \frac{1}{n!N!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-w-W) \times w^n W^N P(w,W,t)dwdW. \quad (9)$$

This transformation can be formally inverted,

$$P(w,W,T) = \exp(w+W)\sum_{n=0}^{\infty}\sum_{N=0}^{\infty}(-1)^{n+N} p(n,N,t)\delta^{(n)}(w)\delta^{(N)}(W), \quad (10)$$

where $\delta^{(n)}(w)$, $\delta^{(N)}(W)$ are derivatives of the Dirac delta function. In analogy with the usual integral transformations, proceeding with the Poisson transformation, we observe that the differential-difference equation will be replaced by a partial differential equation

$$\frac{\partial}{\partial t}P(w,W,t) = L_{\mathrm{att}}^{(\mathrm{IP})}P(w,W,t) + L_{\mathrm{amp}}^{(\mathrm{IP})}P(w,W,t) + L_{\mathrm{nln}}^{(\mathrm{IP})}P(w,W,t), \quad (11)$$

where IP stands for inverse Poisson, and att for attenuation, amp for amplification, nln for a nonlinear process, and

$$L_{\mathrm{att}}^{(\mathrm{IP})}P(w,W,t) = g\frac{\partial}{\partial w}[wP(w,W,t)], \quad (12)$$

$$L_{\mathrm{amp}}^{(\mathrm{IP})}P(w,W,t) = -P\frac{\partial}{\partial W}[wP(w,W,t)]+ \quad (13)$$

$$L_{\mathrm{nln}}^{(\mathrm{IP})}P(w,W,t) = -K\frac{\partial}{\partial w}\left[(w+1)WP(w,W,t)\right]+ K\frac{\partial}{\partial W}\left[(w+1)WP(w,W,t)\right]+ K\frac{\partial^2}{\partial w^2}[wWP(w,W,t)] - K\frac{\partial^2}{\partial w\partial W}[wWP(w,W,t)], \quad (14)$$

with the transformed initial condition (7) in the form

$$P(w,W,t)\big|_{t=t_0} = P_0(w,W,t_0). \quad (15)$$

We find that the terms which decide the partial differential eq. (11) being of the second order are superficially similar to the diffusion terms in the Fokker-Planck equation [10]. The difference consists in that the operator is not elliptic, but hyperbolic. This property is connected to the study of the particle dynamics as well as the absence of an independent parameter, which would enable us to switch diffusion/antidiffusion terms. To get some insight into the behaviour of solutions, we modify somewhat the eq. (11). Let us consider the partial differential equations

$$\frac{\partial}{\partial t}P(w,W,t)=\frac{\partial}{\partial w}\{[gw-k(w+1)W]P(w,W,t)\}$$

$$+\frac{\sigma}{\partial W}\{[-P+K(w+1)W]P(w,W,t)\}+R_j,\ j=1,2 \tag{16}$$

$$R_1=K\frac{\partial^2}{\partial w^2}[wWP(w,W,t)]-K\frac{\partial^2}{\partial w\partial W}[wWP(w,W,t)], \tag{17}$$

$$R_2=0. \tag{18}$$

For $j = 1$, the relation (16) is another form of the relation (11). For $j = 2$, the relation (16) is of first order, which can be solved with the method of characteristics. In connection with this method, we base on the properties of the nonlinear equations

$$\frac{}{dt}w=-gw+K(w+1)W, \tag{19}$$

$$\frac{}{dt}W=P-K(w+1)W, \tag{20}$$

where

$w = w(t)$, $W = W(t)$.

With respect to the unique solvability of these equations, the 'dependence' on the initial condition can be introduced as the solutions of these equations,

$$\frac{}{\partial t}\phi(w,W,t)=-g\phi(w,W,t)$$

$$+K[\phi(w,W,t)+1]\Phi(w,W,t), \tag{21}$$

$$\frac{\partial}{\partial t}\phi(w,W,t)=P-K[\phi(w,W,t)+1]\Phi(w,W,t),$$

$$\times\ \Phi(w,W,t) \tag{22}$$

obeying the initial conditions

$$\phi(w,W,t)|_{t=t_0}=w(t_0)=w_0,$$

$$\Phi(w,W,t)|_{t=t_0}=W(t_0)=W_0. \tag{23}$$

The solution to (16) for $j = 2$ obeying the initial condition (15) is

$$P(w,W,t)=|J|P_0\left(\phi^{-1}(w,W,t),\Phi^{-1}(w,W,t),t_0\right), \tag{24}$$

$$J=\begin{vmatrix}\frac{\partial\phi^{-1}}{\partial w} & \frac{\partial\phi^{-1}}{\partial W}\\ \frac{\partial\Phi^{-1}}{\partial w} & \frac{\partial\Phi^{-1}}{\partial W}\end{vmatrix} \tag{25}$$

and ϕ^{-1}, Φ^{-1} are components of the inverse of the transformation $(\phi,\ \Phi)^{-1}$. Particularly for

$$P_0(w,W,t_0)=\delta(w-w(t_0))\delta(W-W(t_0)), \tag{26}$$

we obtain that

$$P(w,W,t)=\delta(w-w(t))\delta(W-W(t)). \tag{27}$$

The Poisson transform of the solution (27) is a Poisson distribution. To be more precise, we must complete that we have used Poissonian for $p(n,\ N,\ t)$, when the marginal distributions

$$p_1(n,t)=\sum_{n=0}^{\infty}p(n,N,t),\ p_2(N,t)=\sum_{n=0}^{\infty}p(n,N,t), \tag{28}$$

are Poissonian [10] and it holds that

$$p(n,N,t)=p_1(n,t)p_2(N,t). \tag{29}$$

With respect to the transformations (9), (10), we can summarize that when the distribution is Poissonian initially, it is Poissonian for all times. We define steady solutions of (16), as limits for $t_0\to\infty$, if these limits are independent of t. The steady solution may and may not depend on the initial data. For $j = 2$, it can be proved, at least on a stronger assumption formulated below, that the steady solution is

$$P(w,W,t)=\delta\left(w-\frac{P}{g}\right)\delta\left(W-\frac{Pg}{K(P+g)}\right). \tag{30}$$

Here,

$$w=\frac{P}{g},\ W=\frac{Pg}{K(P+g)} \tag{31}$$

are solutions of the algebraic equations

$$0 = -gw + K(w + 1)W, \tag{32}$$

$$P = K(w + 1)W,\ w > 0,\ W > 0. \tag{33}$$

The Poisson transform of the solution (30) is a Poisson distribution, with the Poisson distribution [10] with the parameters given in (31). Considering a small perturbation about the equilibrium point (31), we arrive at the stability matrix

$$\begin{matrix} \dfrac{g^{*}}{P+g} & \dfrac{P+g}{g}, \\ -\dfrac{Pg}{P+g}, & -\dfrac{P+g}{g}K \end{matrix} \tag{34}$$

At the eqs. (32), (33), we arrive in studying the ordinary differential eqs. (19), (20). The stationary solutions (31), are locally asymptotically stable for $t_0 \to \infty$ oo and they can be classified as a node/focus, when (if and only if)

$$\left(\frac{P}{g}+1\right)^2\left(\frac{K}{g}\right)^2 - 2\left(2\frac{P}{g}+1\right)\frac{K}{g} + \frac{1}{\frac{P}{g}+1} \gtrless 0. \tag{35}$$

The local asymptotic stability entails that a restriction of the mapping $(\phi(w,W,t),\ \Phi(w,W,t))$ to a two-dimensional interval Π exists having its centre at the equilibrium point, which is a contracting mapping for $t > t_0$. Hence, it follows that the assertion on the limit (30) holds for any $p_0(w,W,t_0)$ whose support in the distribution or generalized function theory sense is all contained in Π.

Proposition. Let $g > 0$, $P > 0$, $K > 1$, $w_0 \geq 0$, $W_0 \geq 0$. Then the solution of the initial-value problem for eqs. (19), (20), with initial conditions

$$w(t)\big|_{t=t_0} = w(t_0) \equiv w_0,\quad W(t)\big|_{t=t_0} = W(t_0) \equiv W_0, \tag{36}$$

is defined for all $t \geq t_0$ and

$$\lim_{t\to\infty} w(t) = \frac{P}{g},\ \lim_{t\to\infty} W(t) = \frac{P_g}{K(P+g)}. \tag{37}$$

For the proof see Section 4.

For $j = 1$, the second-order differential equation is of the parabolic type, but it is neither heat nor Schrödinger equation, nor any of their generalizations (a Fokker-Planck equation). It is a consequence of the fact that R_1 is not elliptic, but hyperbolic term.

Considering the moments

$$\left\langle \underline{w}^k \underline{W}^l \right\rangle(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^k W^l P(w,W,t)\,dw\,dW, \tag{38}$$

we can rewrite eq. (16), in the form of the hierarchy of equations

$$\frac{d}{dt}\left\langle \underline{w}^k \underline{W}^l \right\rangle(t) = -kg\left\langle \underline{w}^k \underline{W}^l \right\rangle(t)$$

$$+kK\left\langle (\underline{w}+1)\underline{w}^{k-1}\underline{W}^{l+1} \right\rangle(t) + lP\left\langle \underline{w}^k W^{l-1} \right\rangle(t) \tag{39}$$

Particularly,

$$\frac{d}{dt}\langle \underline{w}\rangle(t) = -g\langle \underline{w}\rangle(t)$$

$$+K\left\{\left[\langle \underline{w}\rangle(t)+1\right]\langle \underline{W}\rangle(t) + \langle \Delta\underline{w}\Delta\underline{W}\rangle(t)\right\}, \tag{40}$$

$$\frac{d}{dt}\langle \underline{W}\rangle(t) = P - K\left\{\left[\langle \underline{w}\rangle(t)+1\right]\langle \underline{W}\rangle(t) + \langle \Delta\underline{w}\Delta\underline{W}\rangle(t)\right\}, \tag{41}$$

where

$$\langle \Delta\underline{w}\Delta\underline{W}\rangle(t) = \langle \underline{w}\underline{W}\rangle(t) - \langle \underline{w}\rangle(t)\langle \underline{W}\rangle(t). \tag{42}$$

We consider also

$$\langle \Delta w\rangle^2(t) = \left\langle \underline{w}^2\right\rangle(t) - \langle \underline{w}\rangle^2(t),,$$

$$\left\langle (\Delta\underline{W})^2\right\rangle(t) = \left\langle W^2\right\rangle(t) - \langle \underline{W}\rangle^2(t). \tag{43}$$

It holds that

$$\frac{d}{dt}\left\langle (\Delta\underline{w})^2\right\rangle(t) = -2g\left\langle (\Delta\underline{W})^2\right\rangle(t)$$

$$+2K\left\langle \Delta\underline{w}\Delta[(\underline{w}+1)\underline{W}]\right\rangle(t) + 2K\langle \underline{w}\underline{W}\rangle(t), \tag{44}$$

$$\frac{d}{dt}\left\langle (\Delta\underline{W})^2\right\rangle(t) = -2K\left\langle \Delta[(\underline{w}+1)\underline{W}]\Delta\underline{W}\right\rangle(t), \tag{45}$$

$$\frac{d}{dt}\langle \Delta\underline{w}\Delta\underline{W}\rangle(t) = -g\langle \Delta\underline{w}\Delta\underline{W}\rangle(t)$$

$$+K\left\langle \Delta[(\underline{w}+1)\underline{W}]\Delta\underline{W}\right\rangle(t) - K\langle \underline{w}\underline{W}\rangle(t), \tag{46}$$

where

$$\left\langle \Delta\underline{w}\Delta[(\underline{w}+1)W]\right\rangle(t) = \langle \underline{w}(w+1)\underline{W}\rangle(t)$$

$$-\langle \underline{w}\rangle(t)\langle (\underline{w}+1)\underline{W}\rangle(t), \tag{47}$$

$$\left\langle \Delta[(\underline{w}+1)\underline{W}]\Delta\underline{W}\right\rangle(t)=\left\langle (\underline{w}+1)\underline{W}^2\right\rangle(t)$$

$$-\left\langle(\underline{w}+1)\right\rangle(t)\left\langle\underline{W}^2\right\rangle(t). \tag{48}$$

In physics, the following approximation is made

$$\left\langle \Delta\underline{w}\Delta[(\underline{w}+1)\underline{W}]\right\rangle(t)=\left[\langle\underline{w}\rangle(t)+1\right]\langle\Delta\underline{w}\Delta\underline{W}\rangle(t)$$

$$+\langle\underline{W}\rangle(t)\langle\Delta\underline{w}\Delta\underline{W}\rangle(t), \tag{49}$$

$$\left\langle \Delta[(\underline{w}+1)\Delta\underline{W}]\right\rangle(t)=\left[\langle\underline{w}\rangle(t)+1\right]\left\langle(\Delta\underline{W})^2(t)\Delta\underline{w}\Delta\underline{W}\right\rangle(t). \tag{50}$$

With these approximations, the system of eqs. (40), (41), (44), (45), (46) is closed. A consistency of the approach can be enhanced when the last terms in (40) and (41) are simplified using a lower-order approximation

$$\langle\underline{wW}\rangle(t)\approx\langle\underline{w}\rangle(t)\langle\underline{W}\rangle(t),\quad \langle\Delta\underline{w}\Delta\underline{W}\rangle(t)\approx 0. \tag{51}$$

For an assessment of the approach, we pay attention to the moments

$$\left\langle \underline{n}^k\,\underline{N}^l\right\rangle(t)=\sum_{n=0}^{\infty}\sum_{N=0}^{\infty}n^kN^lp(n,N,t), \tag{52}$$

which can be expressed in terms of moments (38).

Particularly,

$$\langle\underline{n}\rangle(t) = \langle\underline{w}\rangle(t), \tag{53}$$

$$\langle\underline{N}\rangle(t)=\langle\underline{W}\rangle(t), \tag{54}$$

and these 'means' must be nonnegative. Moreover,

$$\left\langle(\Delta\underline{n})^2\right\rangle(t)=\left\langle(\Delta\underline{w})^2\right\rangle(t)+\langle\underline{w}\rangle(t), \tag{55}$$

$$\left\langle(\Delta\underline{N})^2\right\rangle(t)=\left\langle(\Delta\underline{W})^2\right\rangle(t)+\langle\underline{W}\rangle(t), \tag{56}$$

$$\langle\Delta\underline{n}\Delta\underline{N}\rangle(t)=\langle\Delta\underline{w}\Delta\underline{W}\rangle(t). \tag{57}$$

Appropriately arranged, these 'variances' and 'co-variance' must form a positive semidefinite ma-trix. Especially, the variances $\left\langle(\Delta\underline{n})^2\right\rangle(t)$, $\left\langle(\Delta\underline{N})^2\right\rangle(t)$ are nonnegative.

4. Essence of linearization

Another method of solution of eq. (3) which is useful for $g > 0$ and $K = 0$, is the method of characteristic function. The characteristic function of a function $p(n,N,t)$ is defined as follows

$$C^{(n,N)}(is,iS,t)=\sum_{n=0}^{\infty}\sum_{N=0}^{\infty}p(n,N,t)\exp(isn+iSN), \tag{58}$$

which can be inverted as

$$p(n,N,t)=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp(-ins-iNS).$$

$$\times C^{(n,N)}(is,iS,t)ds\,dS. \tag{59}$$

Using the usual Fourier analysis, we can derive that the function (58) obeys the partial differential equation

$$\frac{\partial}{\partial t}C^{(n,N)}(is,iS,t)=L_{\rm att}^{(F)}C^{(n,N)}(is,iS,t)$$

$$+L_{\rm amp}^{(F)}C^{(n,N)}(is,iS,t)+L_{\rm nln}^{(F)}C^{(n,N)}(is,iS,t) \tag{60}$$

and the initial condition

$$C^{(n,N)}(is,iS,t)\Big|_{t=t_0}=C^{(n,N)}(is,iS,t_0), \tag{61}$$

where F stands for Fourier and

$$L_{\rm att}^{(F)}C^{(n,N)}(is,iS,t)=g(e^{is}-1)\frac{\partial}{\partial(is)}C^{(n,N)}(is,iS,t), \tag{62}$$

$$L_{\rm amp}^{(F)}C^{(n,N)}(is,iS,t)=P(e^{is}-1)C^{(n,N)}(is,iS,t), \tag{63}$$

$$L_{\rm nln}^{(F)}C^{(n,N)}(is,iS,t)=K(e^{is-iS}-1)$$

$$\times\left(\frac{\partial}{\partial(is)}+1\right)\frac{\partial}{\partial(iS)}C^{(n,N)}(is,iS,t). \tag{64}$$

The essence of linearization will be evident when it will be applied to equations of motion

$$\frac{d}{dt}\underline{n}=-g\underline{n}+\Gamma+K(\underline{n}+1)\underline{N}+\underline{\Psi}, \tag{65}$$

$$-\frac{d}{dt}\underline{N}=-P+\underline{\Phi}-K(\underline{n}+1)\underline{N}+\underline{\Psi}, \tag{66}$$

with the initial conditions

$$\underline{n}(t)\Big|_{t=t_0}=\underline{n}(t_0)\equiv\underline{n}_0,\quad \underline{N}(t)\Big|_{t=t_0}=\underline{N}(t_0)\equiv\underline{N}_0. \tag{67}$$

where $\underline{n} \equiv \underline{n}(t)$ and $\underline{N} = \underline{N}(t)$ are integer random functions of the time and the joint distribution of integer random variables $\underline{n}_0$ and $\underline{N}_0$ is $p_0(n,N,t_0)$. As the Markovian property and the rate equations (3) describe the stochastic processes $\underline{n}(t)$ and $\underline{N}(t)$ completely, we can (apart from some doubt of the theoretical value of the eqs. (65) and (66)) define processes $\underline{n}_{\rm att}(t)$, $\underline{n}_{\rm nln}(t)$, $\underline{N}_{\rm amp}(t)$, and $\underline{N}_{\rm nln}(t)$ in the same fashion, seeing to the (non-unique) decomposition

$$\underline{n}(t) = \underline{n}_{\rm att}(t) + \underline{n}_{\rm nln}(t),\; \underline{N}(t) = \underline{N}_{\rm amp}(t) + \underline{N}_{\rm nln}(t). \tag{68}$$

Then

$$\underline{\Gamma} = \frac{d}{dt}\underline{n}_{\rm att} + gn,$$

$$\underline{\Phi} = \frac{d}{dt}\underline{N}_{\rm amp} - P,$$

$$\underline{\Psi} = \frac{d}{dt}\underline{n}_{\rm nln} - K(\underline{n}+1)\underline{N} = -\frac{d}{dt}\underline{N}_{\rm nln} - K(\underline{n}+1)\underline{N}. \tag{69}$$

The linearization consists in a replacement of (65), (66) by the equations

$$\frac{d}{dt}\langle \underline{n}\rangle = g\langle \underline{n}\rangle + K\left(\langle \underline{n}\rangle + 1\right)\langle \underline{N}\rangle, \tag{70}$$

$$\frac{d}{dt}\langle \underline{N}\rangle = P - K\left(\langle \underline{n}\rangle + 1\right)\langle \underline{N}\rangle, \tag{71}$$

$$\frac{d}{dt}\delta\underline{n} = -g\delta\underline{n} + \underline{\Gamma} + K\left(\langle \underline{n}\rangle + 1\right)\delta\underline{N} + K\delta\underline{n}\langle \underline{N}\rangle + \underline{\Psi}_{\rm appr}, \tag{72}$$

$$\frac{d}{dt}\delta\underline{N} = \underline{\Phi} - K\left(\langle \underline{n}\rangle + 1\right)\delta\underline{N} + K\delta\underline{n}\langle \underline{N}\rangle + \underline{\Psi}_{\rm appr}, \tag{73}$$

$$\underline{n} = \langle \underline{n}\rangle + \delta\underline{n},\quad \underline{N} = \langle \underline{N}\rangle + \delta\underline{N} \tag{74}$$

where $\underline{\Psi}_{\rm appr}$ is an analogue of $\underline{\Psi}$, and in a change of the initial conditions (67) into

$$\langle \underline{n}\rangle(t)\big|_{t=t_0} = \langle \underline{n}\rangle(t_0) \equiv \langle \underline{n}\rangle_0,$$
$$\langle \underline{N}\rangle(t)\big|_{t=t_0} = \langle \underline{N}\rangle(t_0) \equiv \langle \underline{N}\rangle_0, \tag{75}$$

$$\delta(t)\big|_{t=t_0} = \underline{n}(t_0) - \langle \underline{n}\rangle(t_0),$$
$$\delta\underline{N}(t)\big|_{t=t_0} = \underline{N}(t_0) - \langle \underline{N}\rangle(t_0). \tag{76}$$

Whereas the ordinary differential eqs. (70), (71) are identical with eqs. (19), (20), the eqs. (72), (73) and (74) suggest a linearization of (60), *i.e.*

$$\begin{aligned}\frac{\partial}{\partial t}C^{(\delta n,\delta N)}(is,iS,t) &= \left[g(e^{-is}-1)\frac{\partial}{\partial(is)} + K(e^{is-iS}-1)\langle \underline{N}\rangle\frac{\partial}{\partial(is)}\right]\\ &\times C^{(\delta n,\delta N)}(is,iS,t)\\ &+K(e^{is-iS}-1)\left(\langle \underline{n}\rangle+1\right)\frac{\partial}{\partial(is)}C^{(\delta n,\delta N)}(is,iS,t)\\ &+\Big[g(e^{is}-1)\langle \underline{n}\rangle + P(e^{is}-1-iS)\\ &+K(e^{is-iS}-1-is+iS)\left(\langle \underline{n}\rangle+1\right)\langle \underline{N}\rangle\Big]\\ &\times C^{(\delta n,\delta N)}(is,iS,t)\end{aligned} \tag{77}$$

for the function

$$\begin{aligned}&C^{(\delta n,\delta N)}(is,iS,t)\\ &= \sum_{\delta n=-\langle \underline{n}\rangle}^{\infty}\sum_{\delta N=-\langle \underline{N}\rangle}^{\infty} P\left(\langle \underline{n}\rangle+\delta n, \langle \underline{N}\rangle+\delta N, t\right)\\ &\times \exp(is\delta n + iS\delta N)\\ &= \exp\left(-is\langle \underline{n}\rangle - iS\langle \underline{N}\rangle\right)C^{(n,N)}(is,iS,t)\end{aligned} \tag{78}$$

and the appropriately transformed initial condition (61) is taken into account.

Whereas eq. (16) with $j = 2$ has been truncated rather artificially, the linearization is in the physical community. Similar to the eq. (16), $j = 2$, with the initial condition (15), eq. (77) with an initial condition

$$C^{(\delta n,\delta N)}(is,iS,t)\big|_{t=t_0}\; C^{(\delta n,\delta N)}(is,iS,t) \tag{79}$$

can be solved by the method of characteristics and even that in its pure form. We have performed numerical solution of the initial-value problem (77), in the simplest case of the 'resolved' eqs. (70), (71) *i.e.* for $\langle \underline{n}\rangle(t) = P$, $\langle \underline{N}\rangle(t) = \dfrac{P}{K(P+1)}$. As the initial condition (7), the Poisson distribution has been chosen. In a case when $\langle \underline{n}\rangle(t)$, $\langle \underline{N}\rangle(t)$ were integers, we chose, for comparison,

$p_0(n,N,t_0) = \delta_{n,\langle \underline{n}\rangle(t_0)}\delta_{N,\langle \underline{N}\rangle(t_0)}$. For illustration, the functions $C^{(\delta n,\delta N)}(is,0,t)$, $C^{(\delta n,\delta N)}(0,iS,t)$, $C^{(\delta n,\delta N)}(is,is,t)$, $C^{(\delta n,\delta N)}(is,-is,t)$ have been plotted for $P = 1$, $K = 1$ ($P = 5$, $K = 1/6$), and $t = 10$. For this value, the time dependence has disappeared. For $S = 0$, the distribution of $\delta \underline{n}$, for $s = 0$ the distribution of $\delta \underline{N}$ is characterized, for $S = s$ and $S = -s$, respectively, the distributions of $\delta \underline{n} + \delta \underline{N}$ and of $\delta \underline{n} - \delta \underline{N}$ are characterized. For $P \neq 1$, $K = 1$, $C^{(\delta n,\delta N)}(is,iS,t)$ has jumps which may or may not be connected to the fact that the values of $p(n,N,t)$ determined by this method can be negative, which violates relations (6).

We do not present drawings of moduli and arguments of these functions, but three two-dimensional plots. In Figures 1 and 2 the modulus of characteristic function $C^{(\delta n,\delta N)}(is,iS,t)$ and in Figure 3, its argument can be seen. The behaviour of this function near the point $s = 0$, $S = 0$ enables us to infer the lowest-order moments of $\underline{n}$, $\underline{N}$, $\underline{n} \pm \underline{N}$, especially when a comparison is made with $C^{(\delta n,\delta N)}(is,iS,t_0 = 0)$.

5. Global asymptotic stability

Here, we assume that $g = 1$. In studying the global stability, we have recourse to the phase portrait (see Figure 4) for the equations

$$\dot{x} = K(x-P)\left(y - K(P+1) \right) \quad (80)$$

$$\dot{y} = -K(x-P)\left(y - \frac{P}{K(P+1)} \right). \quad (81)$$

In Figure 4, it can be seen that the sets $\left\{(x,y) : x < P, y = \frac{P}{K(P+1)}\right\}$ and $\left\{(x,y) : x = P, y < \frac{P}{K(P+1)}\right\}$

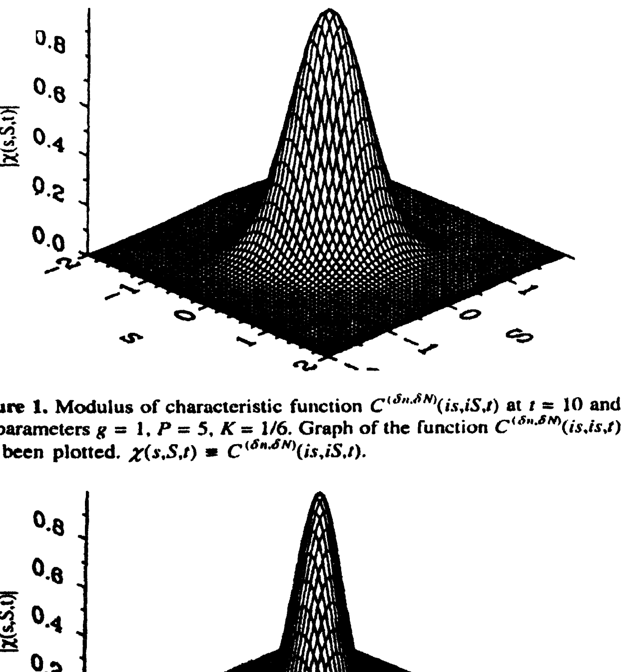

Figure 1. Modulus of characteristic function $C^{(\delta n,\delta N)}(is,iS,t)$ at $t = 10$ and for parameters $g = 1$, $P = 5$, $K = 1/6$. Graph of the function $C^{(\delta n,\delta N)}(is,is,t)$ has been plotted. $\chi(s,S,t) \equiv C^{(\delta n,\delta N)}(is,iS,t)$.

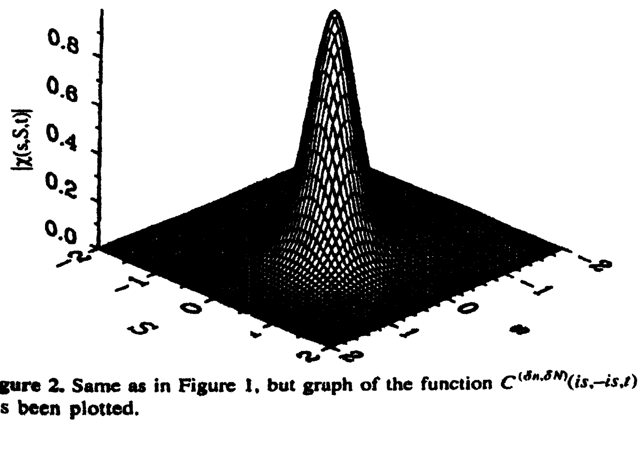

Figure 2. Same as in Figure 1, but graph of the function $C^{(\delta n,\delta N)}(is,-is,t)$ has been plotted.

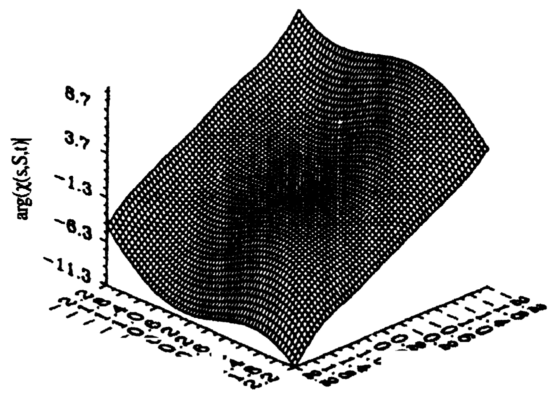

Figure 3. Argument of characteristic function $C^{(\delta n,\delta N)}(is,iS,t)$ at $t = 10$ and for parameters $g = 1$, $P = 5$, $K = 1/6$. The coordinate axes are oriented as in Figure 2 $\chi(s, S, t) \equiv C^{(\delta n,\delta N)}(is,iS,t)$.

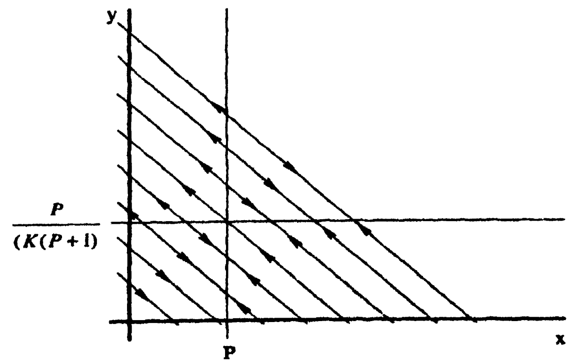

Figure 4. Initial phase portrait.

are attractors and $\left\{(x,y): x < P, y = \frac{P}{K(P+1)}\right\}$ and $\left\{(x,y): x = P, y < \frac{P}{K(P+1)}\right\}$ are repellers.

On passing to the equations

$$\dot{x} = -x + K(x+1)y, \tag{82}$$

$$\dot{y} = P - K(x+1)y, \tag{83}$$

the character of the field is partly changed and partly conserved as indicated in Figure 5.

Let $K > 1$. Then a trapezium $ABCD$ (see Figure 6) can be constructed. We choose A, B, C, D in the form

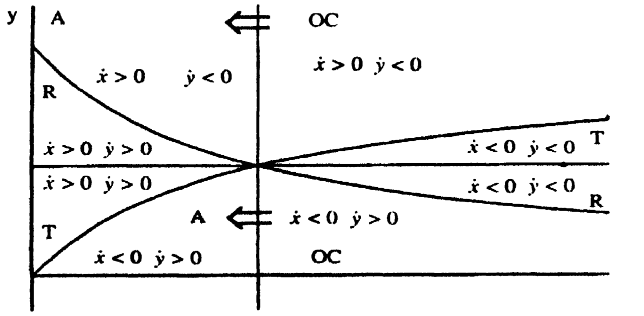

Figure 5. Character of the field. OC-old conserved, R-rotation, T-tendency to equilibrium point, and A-annex.

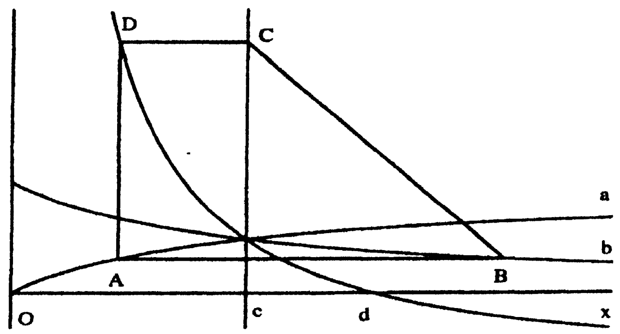

Figure 6. Trapezium useful for the proof of the global convergence.

$$A = \left(\xi, \frac{1}{K}\frac{\xi}{\xi+1}\right), B = \left(P\frac{\xi+1}{\xi} - 1, \frac{1}{K}\frac{\xi}{\xi+1}\right),$$

$$D = \left(\xi, \frac{P}{\xi} - 1 + \frac{1}{K}\frac{\xi}{\xi+1}\right), C = \left(P, \frac{P}{\xi} - 1 + \frac{1}{K}\frac{1}{\xi+1}\right), \tag{84}$$

where the parameter ξ obeys a relation $0 < \xi < P$. The point A lies on the hyperbola a,

$$y = \frac{1}{K}\frac{x}{x+1},$$

the point B lies on the hyperbola b,

$$y = \frac{1}{K}\frac{P}{x+1}, \tag{86}$$

the point C lies on the hyperbola c,

$$x = P, \tag{87}$$

the point D lies on the hyperbola d,

$$y = \frac{P}{x} - 1 + \frac{1}{K}\frac{x}{x+1}. \tag{88}$$

On the side AB, it holds that $\dot{y} \geq 0$, the equality occurring at the point B. On the side CD, $\dot{y} < 0$,. On the side AD, $\dot{x} \geq 0$, the equality occurring at the point A. On the side BC, $(x + y)^{\cdot} \leq 0$, the equality occurring at the point C. We summarize that on the perimeter of the trapezium, the field is directed inside this figure except the points B, A, and C.

At the point $B \in b$, $\ddot{y} > 0$, at the point $A \in a$, $\ddot{x} > 0$, and at the point $C \in c$, $(x+y)^{\cdots} < 0$.

The half-axis $x \geq 0$ can be used as any side AB without B and the half-axis $y \geq 0$ as any side AD without D. Therefore, the quadrant $x \geq 0$, $y \geq 0$ can be added to the family of trapezia under consideration and assigned to $\xi = 0$. Provided that a solution of the eqs. (82), (83) starts at the point $(x(t_0) \geq 0, x(t_0) \geq 0) \neq \left(P, \frac{1}{K}\frac{P}{P+1}\right)$, to each of its points $(x(t), y(t))$, $t \geq t_0$, a trapezium or the whole quadrant $x \geq 0$, $y \geq 0$ can be assigned. Because the parameter ξ is determined by the 'parameters' of the trapezium sides in terms of strictly increasing functions, we can investigate it as increasing function of time for $t \geq t_0$. Since the function $\xi(t)$ is increasing and obeys the inequality $\xi(t) < P$, there exists $\lim_{t\to\infty} \xi(t) \leq P$. Let us suppose for a moment that $\lim_{t\to\infty} \xi(t) = \xi(\infty)$. We define $(d\xi/dt) = a(\xi)$ at all the points of the solution which are not vertices of a trapezium. Let G denote the union of the open intervals included in the set $t \geq t_0$, when the solution is 'near' the vertices B, A, C, *i.e.* let the Lebesgue measure $\lambda(G)$ of this union be finite and its image H after the function $\xi(t)$ be a union of open intervals included in $\{\xi : 0 \leq \xi \leq \xi(\infty)\}$. Then the difference $\langle 0, \xi(\infty)\rangle / H$ is a compact set on which min

$a(\xi) > 0$ can be considered. Let m denote it. We have

$$\xi(\infty) = \int_0^\infty a[\xi(t)]dt > \int_{\langle 0,\infty)/G} mdt = \infty, \tag{89}$$

which is a contradiction.

6. Conclusion

We have analyzed a differential-difference equation for the probability distribution of number of photons and number of excited atoms. Using the Poisson transform, we have obtained the Fokker-Planck equation, which can be solved upon neglect of the diffusion. A second-order equation can be obtained also by the method of characteristic function and it can be solved upon linearization of the underlying physical process.

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